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Open Universes from Finite Radius Bubbles

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The interior of a vacuum bubble in de Sitter space may give an open universe with sufficient homogeneity to agree with observations. Here, previous work by Bucher, Goldhaber and Turok is extended to describe a thin bubble wall with nonzero radius and energy difference across the wall. The vacuum modes present before formation of the bubble propagate into the interior of the open universe and the power spectrum of the resulting gauge invariant gravitational potential is calculated. It appears to become scale invariant on small scales, with onset at about the same scale as that found in the zero radius case. There is sensitivity to the radius and energy difference at large scales, but it is expected that they cannot be strongly constrained because of cosmic variance. The prediction of a scale invariant spectrum seems to be robust with respect to variation of these parameters at small scales, and apparently is a generic feature of the contribution of these modes for these thin wall models.

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I. INTRODUCTION AND BACKGROUND

Inflationary models usually predict $\Omega = 1$ [1], but it has recently been shown they also allow for the possibility of open [2,3,5,4,6] and closed [7] universes. The current models of open universes are either based on nucleation of vacuum bubbles [9,8] or on open universes which are also compact [6]. This paper extends work on models based on bubble nucleation.

Bubble nucleation occurs when a field configuration trapped in a false vacuum tunnels through a barrier to the true vacuum. The tunneling only happens over a finite region of space, so that a bubble of true vacuum forms, with a bubble wall (a domain wall) interpolating between the false vacuum outside and the true vacuum inside. The thickness of the bubble wall and the size of the bubble are determined by parameters in the theory. If there is more than one field, the bubble wall interpolates between true and false vacuum values of all the fields in the theory.

The interior of a vacuum bubble is an open universe [8,9]. If there are random initial conditions before nucleation, the open universe in the bubble interior has primordial inhomogeneities, which may be erased by a subsequent phase of inflation. However, the amount of inflation required to obtain sufficient homogeneity to agree with observation will also make the universe approximately flat, with a critical density $\Omega \approx 1$. Recently proposed [2–5] bubble models accomplish homogeneity without flatness by having three stages. First there is a phase of ordinary inflation, driving $\Omega \to 1$ and erasing initial inhomogeneities. Then a bubble is nucleated, resetting $\Omega \sim 0$ in the interior of the bubble. A period of slow roll inflation then follows, solely for the purpose of driving Ω up to some desired value slightly less than one. At the end of this second stage of inflation, the usual inflationary scenario for reheating takes over. The open universe is the interior of the forward light cone of the center of the bubble. The bubble wall, exterior to this light cone by definition*, has moved out to infinity before some time T in the past.

Modifications using two fields [4,5,7] can result in universes with variable values of Ω. One field provides inflation and the other field resets $\Omega \sim 0$ through bubble nucleation at some point. The advantage of this is that more natural (e.g. polynomial) potentials can be used for the fields. (The one field models need more complicated potentials to resemble the calculable toy models, although these might occur in a supergravity model.) Some care must be taken to avoid large inhomogeneities due to the change of natural time coordinate after tunneling (the synchronicity problem). In addition, predictability can be limited because the amount of inflation, depending on one of the fields, may not necessarily be correlated with the random tunneling time of the other field, producing a range of different Ω's. However, there are modifications which can give more predictability [4,5]. As mentioned in [2], much of the one field model analysis carries over to two field models.

The three stage scenario has some general properties [2]: Before the bubble nucleates, the first stage of inflation drives spacetime to the de Sitter invariant vacuum, the Bunch-Davies vacuum. The Bunch-Davies vacuum was argued in [2] to be the natural vacuum because it has the right symmetries and it goes over to the Minkowski vacuum at short distances. This provides some robustness with respect to initial conditions. The recurrence of inflation after tunneling means the bubble must tunnel from de Sitter space to de Sitter space. The slow roll in the second phase of inflation means that the effective mass after tunneling should be approximately $m^2 \sim 0$. The second stage of inflation must be fine tuned (*i.e.* the parameters in the potential must be fine tuned) within a few percent in order to get $\Omega \sim .1 - .9$.

It is of interest to know how the particulars of the bubble formation affect the resulting density perturbations. Unlike ordinary inflation, here some fine tuning of the parameters in the potential is required to fix Ω, so it is possible that other properties of the resulting theory are sensitive to variation of potential parameters. The focus here is on the imprint of the bubble wall on the vacuum density perturbations. There are also perturbations arising from the fluctuations of the bubble wall itself [4,7,10–12] which must be included.

In order to calculate the density perturbations one does the following: The density perturbations start out (since an initial long period of inflation is assumed) as Bunch-Davies vacuum modes of the false vacuum corresponding to a field of some mass M. Tunneling is formally a process in Euclidean time, so these modes are analytically continued to Euclidean time to give initial conditions. The bubble wall is a boundary between two de Sitter spaces, which will be called the false and true vacua from here on (the final space is not actually a vacuum but the slow rolling second stage of inflation, it will be called the true vacuum here since it is the endpoint of the tunneling). The initial false vacuum fluctuations propagate through the bubble wall and are matched onto a configuration of true vacuum modes using Bogoliubov coefficients. At the nucleation time, the whole system is analytically continued to Lorentzian signature. Using the natural coordinate system for matching across the wall means that the resulting wavefuction must be normalized by hand at some point and can be done here. The perturbations, once inside the forward light

*although for a thick wall there may be some overlap

cone of the bubble (which requires another analytic continuation, since the natural coordinate system is singular on the light cone), are related to perturbations of the gauge invariant gravitational potential, Φ. The power spectrum of Φ, eq. (5.9) along with eqns.(4.6,4.7), is the main result of this paper. It can be combined with contributions from the wall fluctuations and some model of matter content in the universe (*e.g.*, cold dark matter) to get temperature fluctuations to compare to the CMB.

The formalism for fluctuations around a tunneling background has been developed in many papers, see [13–18] and references therein. Bogoliubov coefficients for specific thin and thick bubble walls (both designed to be tractable and yet have enough parameters to be able to test for robustness), were calculated in Minkowski space in [19,11]. In these the fluctuations were massive in both the true and the false vacuum.

The inclusion of gravity and the true vacuum mass $m^2 \approx 0$ add extra complications. In [2], the wall was taken to be a step function, and the radius of the bubble at nucleation, R, was taken to be zero. The power spectrum was calculated for a field with initial false vacuum mass $M^2 = 2H^2$, and the metric perturbations corresponding to the field fluctuations were identified. They also found the matching conditions for a massless field across the light cone, which is subtle due to only half of the modes of a massless field propagating into the light cone. A second paper, [20], generalized this to a field with arbitrary mass in the false vacuum, but still with radius zero. In [18], the quantum state was found for a thin wall bubble with gravity when the fluctuations only have mass on the wall itself.

Effects of perturbations of the wall itself have also been studied in [7,10,12], building on work on bubble fluctuations by [21]. The temperature fluctuations predicted in the CMB from Bunch-Davies vacuum fluctuations (without a bubble) have been calculated in [22] and compared to the CMB measurements and predictions of the conformal [23] vacuum.

In this paper, the thin wall approximation is retained, but the bubble wall is given a finite radius R, for an initial false vacuum mass conformal or larger, $M^2 \geq 2H^2$. Some of the other assumptions used previously are also carried over: that the era preceeding tunneling has resulted in a homogeneous Bunch-Davies vacuum as initial conditions and that bubble nucleation is rare enough that no other bubbles are nearby. First the background solutions, approximations and fluctuation modes are reviewed. After finding Bogoliubov coefficients, the wavefunction is normalized and continued into the forward light cone of the bubble interior (the open universe). Then the power spectrum for the gauge invariant gravitational potential is calculated for this specific background using the results of [2] and compared to earlier cases. These results must be combined with those for fluctuations of the wall itself and some model for structure formation (*e.g.*, cold dark matter) in order to make a prediction for experiments.

Note Added: While this paper was being prepared for submission, the preprint [24] appeared, which independently considers some of the same issues. Some puzzles pointed out in this paper are addressed therein; the final version of this paper has been updated to refer to these where appropriate.

II. BACKGROUND SOLUTION, COORDINATE SYSTEMS, AND APPROXIMATIONS

The background bubble solution for the fluctuations is reviewed first [8,25,18]. The bubble is described to first approximation by classical evolution in Euclidean space. After nucleation, the system is rotated to Lorentzian signature and continues evolving classically [8,25].

The action for a single scalar field ϕ plus gravity is

$$S = \int d^4x\, \sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - (16\pi G)^{-1}R] \tag{2.1}$$

with metric $g_{\mu\nu}$, curvature scalar R, and potential $V(\phi)$. In the flat space case, a solution respecting SO(4) symmetry has minimum Euclidean action, so when gravity is included, one looks for a solution depending only upon the rotationally invariant radial coordinate σ. This has not been proven to be a property of the minimum action solution for the case with gravity, but none with smaller action are known.

With gravity, the Euclideanized metric for tunneling is written in terms of σ^2,

$$ds^2 = d\sigma^2 + b^2(\sigma)[d\tau_E^2 + \cos^2\tau_E d\Omega_{(2)}] \ , \tag{2.2}$$

with $d\Omega_{(2)} = d\theta^2 + \sin^2\theta d\phi^2$.

The Euclidean equations of motion are then

$$\phi''(\sigma) + 3\frac{b'(\sigma)}{b(\sigma)}\phi'(\sigma) = \frac{\partial V}{\partial\phi} \tag{2.3}$$

and

$$\left(\frac{b'(\sigma)}{b(\sigma)}\right)^2 = \frac{1}{b^2(\sigma)} + \frac{8\pi G}{3}\left[\frac{1}{2}\phi'^{\,2} - V[\phi(\sigma)]\right] \ . \tag{2.4}$$

The boundary conditions are

$$\begin{aligned} b(\sigma = 0) = b(\sigma = \sigma_{max}) = 0 \ , \\ \phi'(\sigma = 0) = \phi'(\sigma = \sigma_{max}) = 0. \end{aligned} \tag{2.5}$$

where the turning point in the tunneling solution, σ_{max}, is infinite for the flat space case, but becomes finite once gravity is included [29,26]. For $\phi \approx$ constant, define $H^2 = 8\pi G V(\phi)/3$, so that for de Sitter space, $b(\sigma) = H^{-1}\sin[H\sigma]$. For flat space, $b(\sigma) = \sigma$.

For many of the calculations it will be useful to switch to a second coordinate system, defined by $\tanh u = \cos[H\sigma]$. In terms of these coordinates, the Euclidean line element is:

$$ds^2 = a^2(u)(du^2 + d\tau_E^2 + \cos^2\tau_E d\Omega_{(2)}) \ , \tag{2.6}$$

for de Sitter space,

$$a^2(u) = \frac{1}{H^2\cosh^2 u} \ . \tag{2.7}$$

As one approaches the light cone, $\sigma \to 0$, and $u \to \infty$.

The equations of motion (2.3,2.4) have not been solved in general. A few different approximations are possible, whose validity depends on parameters in $V(\phi)$. A standard approximation is the thin wall approximation, where the energy difference between the true and false vacuum, $\epsilon^2 = H_F^2 - H_T^2$, is very small [8]. The bubble wall, the boundary between the two vacua, is then located approximately at one value of $b(\sigma) = R$. In order to consistently neglect r_b, the width of the wall, it must be small enough compared to other parameters in the theory,

$$r_b \ll R \ , \ \ r_b \ll \epsilon^{-1} \ , \ \ r_b \ll (H_F^2 + H_T^2)^{-1/2} \equiv \Lambda^{-1} \ . \tag{2.8}$$

In order to verify these conditions, the radius R in the presence of gravity can be calculated using the radius R in the absence of gravity, Λ, and ϵ [28].

Another possible approximation is to neglect gravity. The validity of this is measured by the second term in equation (2.4), since for flat Minkowski space $b'(\sigma) = 1$.

In [2] a 'new thin wall' approximation was made. They assumed that $R < H^{-1}$, the wall is thin, the potential is slowly varying after nucleation ($M_{true} \approx 0$), and that the bubble nucleates at about the same energy as that of the false vacuum ($\epsilon \approx 0$). This allows one to treat the background as fixed and to neglect gravity, which they verified is consistent. The energies of the initial and final states are approximately degenerate, $H_F \approx H_T$, and $R \approx 0$. After tunneling, the potential is flat enough for slow roll inflation, so that it is flat enough for the field ϕ to be considered constant. Thus expressions for tunneling from de Sitter space to de Sitter space can be used.

The background for fluctuations in this paper is the thin wall solution found by [18], which, in terms of the coordinate u above, is

$$a(u) = \begin{cases} \frac{1}{H_F\cosh u} & u < u_R \\ \frac{1}{H_F[\cosh(u-u_R)\cosh u_R + \sinh(u-u_R)\sqrt{\cosh^2 u_R - (H_T/H_F)^2}]} & u > u_R \end{cases} \tag{2.9}$$

$$\phi_b(u) = \begin{cases} \phi_i & \text{if } u < u_R \\ \phi_n & \text{if } u > u_R \end{cases} \tag{2.10}$$

This solution requires $u_R > 0$. The interior $a(u)$ can be recognized as $(H_T\cosh(u+\delta))^{-1}$, the metric for a shifted value of u, so that the metrics are matched at the same value of $a(u)$, which is the radius

$$R = \frac{1}{H_F\cosh u_R} = \frac{1}{H_T\cosh(u_R+\delta)} \ . \tag{2.11}$$

The coordinate shift δ obeys

$$e^\delta = \frac{(1-\sqrt{1-(H_F R)^2})(1+\sqrt{1-(H_T R)^2})}{(H_T R)(H_F R)} \tag{2.12}$$

Setting the $H_T = H_F$ takes $\delta \to 0$. Taking the matching radius $R \to 0$ means (from eq. (2.11)) $H_T \to H_F$ and $u_R \to \infty$.

Once the bubble is nucleated, one rotates to a Lorentzian signature to continue the evolution. Usually the nucleation time is chosen at $\tau_E = 0$. The coordinate system for the Euclidean description of the bubble can be analytically continued,

$$\begin{aligned} ds^2 &= d\sigma^2 + b^2(\sigma)[-d\tau^2 + \cosh^2 \tau d\Omega_{(2)}] \\ &= \frac{1}{H^2 \cosh^2 u}[du^2 - d\tau^2 + \cosh^2 \tau d\Omega_{(2)}] \end{aligned} \tag{2.13}$$

The equations of motion for the σ dependent background remain unchanged.

On the light cone of the bubble center, $b(\sigma) = 0$ ($u = \infty$), so there is a coordinate singularity. One can continue to the interior of the light cone by rotating, $\sigma = -iT, \tau = \chi - i\pi/2, b = -ic$, to the coordinate system with line element

$$ds^2 = -dT^2 + c^2(T)[d\chi^2 + \sinh^2 \chi d\Omega_{(2)}] \ . \tag{2.14}$$

The equations of motion are now

$$\begin{aligned} &\ddot{c}(T) + 3\frac{\dot{c}(T)}{c(T)}\dot{\phi}(T) = -\frac{\partial V}{\partial \phi} \\ &(\frac{\dot{c}(T)}{c(T)})^2 = \frac{1}{c^2(T)} + \frac{8\pi G}{3}[\frac{1}{2}\dot{\phi}^2 + V[\phi(T)]] \ . \end{aligned} \tag{2.15}$$

For de Sitter space, the background solution except at the wall (which is not inside the interior of the light cone for a thin wall) is $c(T) = H^{-1}\sinh[HT]$.

Classically, then, the following picture arises [8]: there are two regions of space considered here, outside the light cone of the bubble center (which will be called region II, as was done in [2]) and inside the light cone of the bubble center (region I).† For the thin and new thin wall approximation, the bubble wall is approximately located at just one value of $b(\sigma) = R$. The wall moves exterior to the light cone of the bubble's center, approaching it on the curve $b(\sigma)^2 = R^2$. (For $R = 0$ the bubble wall is the light cone.) The bubble wall has no intersection with any of the curves in region I. There may be some overlap with region I for a wall that isn't thin. The energy difference between the true and false vacuum goes into accelerating the wall (which as it increases in size, due to surface tension, contains more energy).

In region I, inside the future light cone, the SO(3,1) invariant line element T^2 behaves just like time in the equations of motion. The field ϕ is constant on lines of constant T and thus region I can be viewed as an open Freedman-Robertson Walker universe with scale factor $c(T)$. The constant T surfaces are not Cauchy surfaces, however, which will be important later on when the spectrum is considered. Since a thin bubble wall is confined to region II, it occurs "before" $T = 0$ for the open universe.

There are consistency requirements for parameters in a given background theory, some of which are automatically satisfied in the thin wall approximation. The action of the bubble should be large in order for the semiclassical approximation to be valid, and can be calculated directly from the action in the absence of gravity [28]. A large action makes the nucleation rate small, another requirement for the scenario here, so that tunneling occurs only after inflation has driven the initial conditions into the Bunch-Davies vacuum. The surface tension controls how much the bubble wall remains close to its classical configuration, and thus it must be large for the classical solution to be a good saddle point. Increasing the surface tension increases the action of the bubble, so these two conditions are related. If the false vacuum mass is too small, the lowest action solution is the Hawking Moss solution, [29] corresponding to stochastic fluctuations over the top of the barrier rather than tunneling through it. This gives unacceptable density perturbations, [26,27,30].

It is not clear how difficult it is to construct potentials which can give rise to the general thin wall background used here. Simple ϕ^4 potentials in one field models tend to result in the Hawking Moss instanton [7] when the slow roll condition in the true vacuum is combined with thin wall conditions. A linear potential in both the true and false vacuum (triangular barrier) apparently forces the energy difference to be negligible. In addition to finding out which parameter values are consistent with the thin wall approximation, it is particularly interesting to see what potentials arise naturally in particle physics models.

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†To cover de Sitter space requires another region antipodal to I, but it will only be relevant here when verifying the wavefunction normalization inside region I, summarized in section 5.

III. FLUCTUATIONS AROUND THE BACKGROUND SOLUTION

With the background solution of the previous section, eq. (2.9), one can now solve for basis functions in both the true and false vacuum and match the false vacuum fluctuations across the wall. (A rigorous foundation for this using the WKB formalism is found in [17]. See also [13–16,18] and references therein.)

The potential right before nucleating the bubble is assumed to be a deep well, and after nucleating practically flat, for the second bout of inflation. Schematically,

$$V(\phi) = \begin{cases} V_F + M^2(\phi-\phi_i)^2 + \cdots & \phi \approx \phi_i \\ -\mu^3\phi & \phi \approx \phi_n \end{cases} \tag{3.1}$$

For a thin wall, the matching is in region II. The discussion of basis functions as in [2] is reviewed first. The region II fluctuations around the classical solution will be given for Lorentzian signature and then rotated to Euclidean signature. They obey (since the background is constant except at the wall)

$$(\Box - \frac{\delta^2 V(\phi)}{\delta\phi^2}|_{\phi=\phi_b})\delta\phi = (\Box - m^2(\sigma))\delta\phi = 0 \tag{3.2}$$

The mass only depends upon σ because the background solution $\{\phi_b(\sigma), b(\sigma)\}$ only depends upon σ. So with a step function at $b(\sigma) = R$, the boundary conditions are

$$m^2(\sigma) = \begin{cases} M^2 \ , & R < b(\sigma) < \infty \\ 0 \ , & 0 < b(\sigma) < R \end{cases} \tag{3.3}$$

Expanding eq. (3.2) in terms of $(\sigma, \tau, \theta, \phi)$, and the background de Sitter metric,

$$[\frac{\partial^2}{\partial\sigma^2} + 3H\cot H\sigma\frac{\partial}{\partial\sigma} - \frac{H^2}{\sin^2 H\sigma}(\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} + \frac{\mathbf{L}^2}{\cosh^2\tau}) - m^2(\sigma)]\phi(\sigma,\tau,\theta,\phi) = 0. \tag{3.4}$$

The eigenfunctions of the Laplacian on the 3d hyperboloid satisfy

$$(\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} + \frac{\mathbf{L}^2}{\cosh^2\tau})Y_{plm}(\tau,\theta,\phi) = -(1+p^2)Y_{plm}(\tau,\theta,\phi) \ , \tag{3.5}$$

so schematically

$$\phi(\sigma,\tau,\theta,\phi) = \int dp\, S_p^{\nu'}(\sigma)Y_{plm}(\tau,\theta,\phi) \ . \tag{3.6}$$

Here $Y_{plm}(\tau,\theta,\phi) = f_{pl}(\tau)Y_{lm}(\theta,\phi)$. The $Y_{lm}(\theta,\phi)$ are the usual spherical harmonics. Since all the mass dependence is in $S_p^{\nu'}(\sigma)$, through

$$\nu' = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} - \frac{1}{2} \ , \tag{3.7}$$

consider σ dependence first.

The equation of motion for $S_p^{\nu'}(\sigma)$ is

$$[\frac{\partial^2}{\partial\sigma^2} + 3H\cot H\sigma\frac{\partial}{\partial\sigma} + \frac{H^2(1+p^2)}{\sin^2 H\sigma} - m^2(\sigma)]S_p^{\nu'}(\sigma) = 0. \tag{3.8}$$

The most convenient basis, different from the one usually used to describe the tunneling as in [8], is the (u,τ) coordinate system used to describe the instanton in [18]. Recalling that $\cos[H\sigma] = \tanh u$,

$$S_p^{\nu'}(\cos[H\sigma]) = \frac{F_{\nu'}^{ip}(u)}{a(u)} \tag{3.9}$$

where

$$
\begin{aligned}
F^{ip}_{\nu'}(u) = &\ \tfrac{1}{4\pi\sqrt{|p|}}\{a_{\nu'+}\Gamma(1-ip)P^{+ip}_{\nu'}(\tanh u) - \tfrac{C_2}{|C_2|}a_{\nu'-}\Gamma(1+ip)P^{-ip}_{\nu'}(\tanh u)\} \\
&=_{x\equiv\tanh u}\ \tfrac{1}{4\pi\sqrt{|p|}}\{a_{\nu'+}(\tfrac{1+x}{1-x})^{\frac{ip}{2}}\,{}_2F_1(-\nu',\nu'+1;1-ip;\tfrac{1-x}{2}) \\
&-\tfrac{C_2}{|C_2|}a_{\nu'-}(\tfrac{1+x}{1-x})^{\frac{-ip}{2}}\,{}_2F_1(-\nu',\nu'+1;1+ip;\tfrac{1-x}{2})\}\ .
\end{aligned}
\tag{3.10}
$$

The coefficients

$$
a_{\nu'\pm} = \sqrt{\frac{1\pm\sqrt{1-|C_2|^2/C_1^2}}{2}}
\tag{3.11}
$$

depend on momentum,

$$
\begin{aligned}
&C_1(p) = 2\pi[1+\tfrac{\sin^2[\pi\nu']}{\sinh^2[\pi p]}] \\
&C_2(p) = 2\pi\tfrac{\Gamma(1-ip)\Gamma(ip-\nu')}{\Gamma(1+ip)\Gamma(-ip-\nu')}\tfrac{\sin[\pi\nu']\sin[\pi(\nu'-ip)]}{\sinh^2[\pi p]} \\
&= -2\pi^2\tfrac{\Gamma(1-ip)}{\Gamma(1+ip)}\tfrac{\sin\pi\nu'}{\sinh^2\pi p}\tfrac{1}{\Gamma(-ip-\nu')\Gamma(1-ip+\nu')}
\end{aligned}
\tag{3.12}
$$

The coefficient $C_1(p)$ is real for values of ν' considered here, $C_2(-p) = C_2^*(p)$ and $\lim_{p\to\infty,\nu'\ fixed} C_2(p) = 0$.

The functions $F^{ip}_{\nu'}$ are normalized as

$$
\int_{-\infty}^{\infty} du F^{ip}_{\nu'}(u)F^{ip'*}_{\nu'}(u) = \frac{1}{8\pi|p|}\delta(p-p')\ .
\tag{3.13}
$$

There is a continuum of modes for $0 \le p \le \infty$. (The τ dependence introduces some more p dependence so that the full wavefunctions are nonsingular at $p = 0$.)

For the special cases of $\nu' = 0$ (conformal mass, $M^2 = 2H^2$) and $\nu' = 1$ ($M^2 = 0$ as is the case after tunneling), one has

$$
\begin{aligned}
F^{ip}_0 &= \tfrac{e^{ipu}}{4\pi\sqrt{|p|}} \\
F^{ip}_1 &= \tfrac{e^{ipu}}{4\pi\sqrt{|p|}}\tfrac{(\tanh u-ip)}{1-ip}
\end{aligned}
\tag{3.14}
$$

When $0 \le \nu' \le 1$ (ie when $M^2/2H^2 < 1$), there is an additional normalizable state which was shown to be part of the basis of states in [31]. They showed that it is necessary to include this state in order to obtain the proper Wightman function for the Bunch-Davies vacuum of de Sitter space.

The Wightman function is the sum over positive frequency modes times the negative frequency modes (a vacuum dependent assignment),

$$
\begin{aligned}
G^+(x,x') &= \langle 0|\hat\phi(x)\hat\phi(x')|0\rangle \\
&= \langle 0|\textstyle\sum_{k,k'}\{(u_{+,k}(x)\hat a(k) + u_{-,k}(x)\hat a^\dagger(k))\} \\
&\quad \times\{(u_{+,k'}(x')\hat a(k') + u_{-,k'}(x')\hat a^\dagger(k'))\}|0\rangle \\
&= \textstyle\sum_k u_{+,k}(x)u_{-,k}(x')\ .
\end{aligned}
\tag{3.15}
$$

The modes $u_{\pm,k}(x)$ obey the Klein Gordon equation and are normalized according to

$$
\begin{aligned}
(u_{+,k}(x),u_{+,k'}(x)) &= \delta(k-k') \\
(u_{-,k}(x),u_{-,k'}(x)) &= -\delta(k-k') \\
(u_{+,k}(x),u_{-,k'}(x)) &= 0\ .
\end{aligned}
\tag{3.16}
$$

where the inner product is the Klein Gordon inner product:

$$
(\phi_1(x),\phi_2(x)) = -i\int_\Sigma d\Sigma^\mu[-g_\Sigma(x)]^{\frac{1}{2}}\{\phi_1(x)\partial_\mu\phi_2^*(x) - \partial_\mu\phi_1(x)\phi_2^*(x)\}\ .
\tag{3.17}
$$

The integration is over Σ, a (globally) Cauchy surface, *i.e.*, a spacelike hypersurface which every non-spacelike curve intersects exactly once [32].

The additional normalizable state for $0 < \nu' \le 1$ has $p = i\nu'$,

$$
F^{-\nu'}_{\nu'} = \frac{1}{4\pi\sqrt{|p|}}(\cosh u)^{-\nu'} \quad 0 < \nu' \le 1
\tag{3.18}
$$

It is for a discrete value of p and hence corresponds to a state which normalizes to a constant times a Kronecker delta function, not a Dirac δ function.

There is a discussion of the properties of these states in [31]. When continued to the inside of the light cone of the bubble center, region I, these states appear as supercurvature modes which are naively unnormalizable. As explained in [31], this lack of normalizability is due to the curves of fixed time T failing to be Cauchy surfaces. (The light cone of the bubble center, a non-spacelike curve, is $T = 0$, and never intersects curves with $T \neq 0$.) They show that for the other modes, with $p^2 > 0$, the inner products on curves of fixed time T coincide with their norms on Cauchy surfaces because of sufficiently fast falloff at infinity. The supercurvature modes are normalizable on a Cauchy surface, for example along the $\tau = 0$ surface outside the forward light cone [31], and the result is independent of Cauchy surface as it should be.

The $\ell = 0$ mode for $\nu' \to 1$ has zero norm, as shown in [31], perhaps due to overlap with a translational zero mode in the system, similar to what happens in [12]. Aside from this case, the normalizable discrete state with special $p = i\nu'$ occurs as part of the Bunch-Davies vacuum for all ℓ, m, when $0 < \nu' \leq 1$.

When τ dependence is included, the Wightman function can be used, via the Klein Gordon inner product, to project upon positive or negative frequency modes for a given set of basis functions. This was used in [2,20] to identify the $\ell = 0$ positive freqency mode functions for the Bunch Davies vacuum

$$\phi^+_{\pm p}(u,\tau) = H \cosh u F^{\pm ip}_{\nu'}(u) \frac{(e^{\frac{\pi p}{2}} e^{-ip\tau} - e^{\frac{-\pi p}{2}} e^{ip\tau})}{\sqrt{e^{\pi p} - e^{-\pi p}} \cosh \tau} \quad 0 < p < \infty \tag{3.19}$$

with corresponding negative frequency modes

$$\phi^-_{\pm p}(u,\tau) = H \cosh u F^{\pm ip}_{\nu'}(u) \frac{(e^{\frac{-\pi p}{2}} e^{-ip\tau} - e^{\frac{\pi p}{2}} e^{ip\tau})}{\sqrt{e^{\pi p} - e^{-\pi p}} \cosh \tau} \quad 0 < p < \infty \ . \tag{3.20}$$

For general ℓ, the τ dependent factor $f_{pl}(\tau)$ is proportional to (from [31]):

$$\begin{aligned} f_{pl}(\tau) \ &\propto \frac{\Gamma(ip+l+1)}{\Gamma(ip)} \frac{1}{\sqrt{\sinh(\tau+\pi i/2)}} P^{-l-1/2}_{ip-1/2}(\cosh(\tau+\pi i/2)) \\ &= i(-1)^{l+1} \sqrt{\tfrac{2}{\pi}} p \frac{\Gamma(-ip)}{\Gamma(-ip+l+1)} \cosh^l \tau \left(\frac{d}{d\sinh\tau}\right)^l \frac{\sin p(\tau+i\pi/2)}{\cosh \tau} \ . \end{aligned} \tag{3.21}$$

For simplicity, the discussion will be restricted hereon to $\ell = m = 0$, since the matching across the bubble wall depends on σ and not τ. So, suppressing ℓ, m dependence, the operator describing fluctuations is

$$\begin{aligned} \delta\hat{\phi}(u,\tau) = \ & \int_0^\infty dp \{\phi^+_p(u,\tau)\hat{a}(p) + \phi^+_{-p}(u,\tau)\hat{a}(-p) + h.c.\} \\ & + \sum_{p_{disc}} \phi^+_{p_{disc}}(u,\tau)\hat{a}(p_{disc}) + h.c. \end{aligned} \tag{3.22}$$

with the $\phi^\pm_{\pm p}$ normalized in the Klein-Gordon inner product, eqn (3.17).

IV. INITIAL CONDITIONS, MATCHING ACROSS WALL AND NORMALIZATION

Given the positive frequency modes in the false vacuum, the next step is to analytically continue to Euclidean time and match them on to the initial conditions for fluctuations around the instanton, eq. (2.9). However, the initial conditions for the instanton are not directly visible in the (σ, τ_E) or (u, τ_E) coordinate system. In these coordinates, the bubble wall is present for all time τ_E, so there is no false vacuum at any time τ_E. However, these coordinates are useful since a constant value of $b(\sigma), a(u)$ defines the bubble wall.

This problem is emphasized in [18], where another coordinate system is suggested which maps $(\sigma, \tau_E) \to (\tilde{\sigma}, \tilde{\tau}_E)$. The new time coordinate $\tilde{\tau}_E$ coincides with τ_E along the line $\tau_E = 0$, the nucleation time, where analytic continuation to Lorentzian signature takes place. The $(\tilde{\sigma}, \tilde{\tau}_E)$ coordinate system can be used to specify initial conditions, but it is not useful in practice for matching across the bubble wall. At early times $\tilde{\tau}_E$ in the $(\tilde{\sigma}, \tilde{\tau}_E)$ coordinate system, the bubble wall has not yet formed, so that the vacuum can be identified. One can do the matching using either coordinate system. The modes corresponding to the false vacuum Bunch-Davies modes in the (σ, τ_E) coordinates coincide with those in the far past, at early times $\tilde{\tau}_E$, in the coordinate system which has the proper initial conditions (false Bunch-Davies vacuum). So, knowing that the $(\tilde{\sigma}, \tilde{\tau})$ coordinate system exists justifies starting with false vacuum Bunch-Davies modes exterior to the bubble, as these are continued from modes at early $\tilde{\tau}_E$.[‡]

[‡]I thank Arley Anderson for discussions about this.

An important consequence of not specifying the initial conditions at fixed τ_E (the bubble wall exists for all τ_E, the initial condition is without the bubble), that is, using the basis functions eq. (3.19) for matching, is that the resulting wavefunction matched across the bubble wall is not automatically normalized. So the wavefunction is normalized after the nucleation time $\tau_E = 0$. The Minkowski space analogue of this was done in, *e.g.*, [19].

With this understood, the first step is to analytically continue the positive frequency wavefunctions (3.19) of the false vacuum to Euclidean time, $\tau \to i\tau_E$,

$$A_{p,\nu'}(u,\tau_E) = \phi^+_{\pm p}(u,\tau_E) = H\cosh u F^{\pm ip}_{\nu'}(u)\frac{(e^{\frac{\pi p}{2}}e^{p\tau_E} - e^{\frac{-\pi p}{2}}e^{-p\tau_E})}{\sqrt{e^{\pi p} - e^{-\pi p}}\cos\tau_E} \quad . \tag{4.1}$$

Recall $\nu' = \sqrt{\frac{9}{4} - M^2} - \frac{1}{2}$, and $F^{ip}_{\nu'}(u)$ is given in eq. (3.10). These are to be matched at $u = u_R$ onto linear combinations of modes corresponding to the true vacuum, where $M^2 \approx 0$, that is $\nu' \sim 1$. The basis functions for true vacuum modes are

$$B_{\pm p}(u,\tau_E) = H_T\cosh(u+\delta)F^{\pm ip}_1(u+\delta)\frac{(e^{\frac{\pi p}{2}}e^{p\tau_E} - e^{\frac{-\pi p}{2}}e^{-p\tau_E})}{2\sqrt{e^{\pi|p|} - e^{-\pi|p|}}\cos\tau_E} \quad . \tag{4.2}$$

where $F^{ip}_1(u)$ is given in eq. (3.14).

Because the τ_E dependence is continuous, one can only match modes with equal values of $|p|$ (the matching has to be true for all times, so the time dependence has to be the same on both sides). Both $B_{\pm p}(u,\tau_E)$ are positive frequency modes, so this does not look like particle creation. Continued inside the light cone, where the space and time dependence get interchanged, it can be interpreted as causing particle creation. Matching the functions and their first derivatives at $u = u_R$,

$$\begin{aligned} A_p(u,\tau_E) &= \alpha_p B_p(u,\tau_E) + \beta_p B_{-p}(u,\tau_E) \\ \partial_u A_p(u,\tau_E) &= \alpha_p\partial_u B_p(u,\tau_E) + \beta_p\partial_u B_{-p}(u,\tau_E)\ , \end{aligned} \tag{4.3}$$

gives

$$\begin{aligned} \alpha_p &= \tfrac{\partial_u B_{-p}A_p - B_{-p}\partial_u A_p}{B_p\partial_u B_{-p} - B_{-p}\partial_u B_p} \\ \beta_p &= \tfrac{-\partial_u B_p A_p + B_p\partial_u A_p}{B_p\partial_u B_{-p} - B_{-p}\partial_u B_p} \end{aligned} \tag{4.4}$$

evaluated at $u = u_R$.

So for false vacuum mass $M^2 = 2H^2$, (using the explicit form of $F^{ip}_0(u)$ in eq. (3.14)),

$$\begin{aligned} \alpha_p &= \tfrac{e^{-ip\delta}}{2p(p-i)}[2p^2 + \tfrac{\cosh\delta - ip\sinh(2u_R+\delta)}{\cosh u_R\cosh(u_R+\delta)}] \\ \beta_p &= \tfrac{e^{ip(2u_R+\delta)}}{2p(p+i)}\tfrac{\cosh\delta - ip\sinh\delta}{\cosh u_R\cosh(u_R+\delta)} \end{aligned} \tag{4.5}$$

For more general masses in the false vacuum, the Bogoliubov coefficients are:

$$\begin{aligned} \alpha_{p,\nu'} &= \tfrac{e^{-ipu_R}}{2p(p-i)}\{(1+p^2-(\nu'+2)\tanh u_R(\tanh(u_R+\delta)+ip))4\pi\sqrt{|p|}F^{ip}_{\nu'}(u_R) \\ &+(\nu'-ip+1)(\tanh(u_R+\delta)+ip)a_{\nu'+}\Gamma(1-ip)P^{ip}_{\nu'+1}(\tanh u_R) \\ &-(\nu'+ip+1)(\tanh(u_R+\delta)+ip)\tfrac{C_2}{|C_2|}a_{\nu'-}\Gamma(1+ip)P^{-ip}_{\nu'+1}(\tanh u_R)\} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \beta_{p,\nu'} &= \tfrac{e^{ip(u_R+\delta)}}{2p(p+i)}\{(1+p^2-(\nu'+2)\tanh u_R(\tanh(u_R+\delta)-ip))4\pi\sqrt{|p|}F^{ip}_{\nu'}(u_R) \\ &+(\nu'-ip+1)(\tanh(u_R+\delta)-ip)a_{\nu'+}\Gamma(1-ip)P^{ip}_{\nu'+1}(\tanh u_R) \\ &-(\nu'+ip+1)(\tanh(u_R+\delta)-ip)\tfrac{C_2}{|C_2|}a_{\nu'-}\Gamma(1+ip)P^{-ip}_{\nu'+1}(\tanh u_R)\}\ . \end{aligned} \tag{4.7}$$

One can verify that as the radius goes to zero, $u \to \infty, \delta \to 0$, and these reduce to the expressions of [2,20],

$$\alpha_p \to_{R\to 0} a_{\nu'+}\ , \quad \beta_p \to_{R\to 0} -\frac{C_2}{|C_2|}a_{\nu'-} \tag{4.8}$$

In showing this it is helpful to recall that $\lim_{u\to\infty}\Gamma(1-ip)P^{ip}_{\nu'}(\tanh u) \sim e^{ipu}$, so that terms cancel between $F^{ip}_{\nu'}$ and $P^{ip}_{\nu'+1}$. In addition, as $p \to \infty$ these go over to the conformally coupled $M^2 = 2H^2$ case,

$$\alpha_{p,\nu'} \to 1 \, , \beta_{p,\nu'} \to 0 \, . \tag{4.9}$$

That is, at short distances, the dependence upon the scales set by the mass and radius drops out.

These Bogoliubov coefficients are for p real. When $0 < \nu' \leq 1$, as mentioned earlier, there are also states corresponding to supercurvature modes in the false vacuum. These supercurvature modes have τ dependence which match onto eigenfunctions of the Laplacian with u dependence

$$\begin{array}{l} F_1^{\nu'}(u+\delta) = e^{-\nu'(u+\delta)} \frac{(\tanh(u+\delta)+\nu')}{1+\nu'} \\ or\ F_1^{-\nu'}(u+\delta) = e^{\nu'(u+\delta)} \frac{(\tanh(u+\delta)-\nu')}{1-\nu'} \quad , \end{array} \tag{4.10}$$

non-normalizable on any surface. The first eigenfunction is well behaved as $u \to \infty$ and can be connected across the wall at $u = u_R$ to eigenfunctions which are well behaved at $u = -\infty$, however, it is not possible generically to match the original vacuum mode and its first derivative (two constraints) across the wall without the second eigenfunction which blows up at $u_R = \infty$. The power spectrum inside the bubble may be computed anyhow, *e.g.* as is shown in [33] for general supercurvature modes, but it is possible that the lack of normalizability means that there is some instability (and thus inconsistency in this naive treatment). The way out of this is [24] to not include the state which has become unnormalizable due to the wall, but to look for any new states that may have become normalizable at the same time, again due to matching across the wall. Thus one has an added degree of freedom (the value of p) which can be used to try to satisfy the matching constraints. As they show, a normalizable supercurvature state is present for $M^2/H^2 \leq 1$ for $R = 0$, and for larger values of M when $R \neq 0$.

Similarly, there is also a supercurvature mode present in the 'true' vacuum of the bubble interior, where $\nu' \approx 1$. It does not have the same time dependence as any state in the false vacuum, and so is not excited by the continuation of the false vacuum fluctuations across the wall. It is present in some cases, *e.g.* if there is no bubble, [31,22], or if the field providing density fluctuations is massless on both sides of the bubble wall [3]. If the mass of the field in the false vacuum is large enough, the question of these supercurvature modes does not occur. These modes will not be considered further here.

Once the modes have been matched across the wall, the wavefunctions are analytically continued to Lorentzian signature at $\tau_E = 0 = -i\tau$ to give the wavefunctions for a bubble outside of the light cone of the center of the bubble. This is straightforward to do by substituting $\tau = i\tau_E$.

The resulting wavefunctions are

$$\phi^{+}_{+p,\mathrm{unnorm}}(u,\tau) = \theta(u_R - u) A_{p,\nu'}(u,\tau) + \theta(u - u_R)(\alpha_p B_p(u,\tau) + \beta_p B_{-p}(u,\tau)) \, , \tag{4.11}$$

where $\theta(u)$ is a step function. As mentioned before, $\phi^{+}_{+p,\mathrm{unnorm}}(u,\tau)$ unnormalized. In fact, it is not even[§] orthogonal to $\phi^{+}_{-p,\mathrm{unnorm}}(u,\tau)$. This can be remedied by using the same method in [20].

As the τ dependence is the same for both fields, and corresponds to a properly normalized wavefunction (as would be automatic if there were no bubble wall, which also has the same τ dependence), it can be separated out first,

$$\phi^{+}_{+p,\mathrm{unnorm}}(u,\tau) = \frac{\mathcal{F}^{+p}(u)}{a(u)} \frac{(e^{\frac{\pi p}{2}} e^{p\tau_E} - e^{\frac{-\pi p}{2}} e^{-p\tau_E})}{2\sqrt{e^{\pi|p|} - e^{-\pi|p|}} \cos\tau_E} \quad . \tag{4.12}$$

Then, the $\mathcal{F}^{\pm p}(u)$ are eigenfunctions of a self-adjoint operator with eigenvalue depending on p^2, and so eigenfunctions with different eigenvalues are orthogonal,

$$\int_{-\infty}^{\infty} du \mathcal{F}^{+p}(u) \mathcal{F}^{+p'}(u) = \frac{D_1(p)}{8\pi|p|} \delta(p+p') + \frac{D_2(p)}{8\pi|p|} \delta(p-p') \tag{4.13}$$

The coefficients $D_1(p), D_2(p)$ can be read off from the asymptotic behavior at $u \to \pm\infty$. After some algebra they are seen to be:

$$\begin{array}{rl} D_1(p) & = \frac{1}{2}\left[|\alpha_p|^2 + |\beta_p|^2 + 1\right] = D_1(-p) \\ D_2(p) & = \left[\alpha_p \beta_p + \frac{C_2}{2C_1}\right] = D_2(-p)^* \end{array} \tag{4.14}$$

Then the normalized wavefunction is

[§] I thank M. Bucher for pointing this out and for suggesting the method used below.

$$\phi^+_{+p}(u,\tau) = b_{+,p}(u_R)\phi^+_{+p,\mathrm{unnorm}}(u,\tau) - \frac{D_2(p)}{|D_2|} b_{-,p}(u_R)\phi^+_{-p,\mathrm{unnorm}}(u,\tau) \tag{4.15}$$

where

$$b_{\pm,p}(u_R) = \sqrt{\frac{D_1}{D_1^2 - |D_2|^2}} \sqrt{\frac{1 \pm \sqrt{1 - |D_2|^2/D_1^2}}{2}} \ . \tag{4.16}$$

This normalization can be checked by a different calculation once the modes are continued into the future light cone of the bubble center, as described below. The other method of normalization has no integration over (the analytic continuation of) u, and so does not depend on continuity the same way. The two calculations agree. So the operator describing fluctuations, $\delta\hat{\phi}$, is as in eq. (3.22), with $\phi^+_p(u,\tau)$ given in eq. (4.15).

V. CONTINUATION INTO OPEN UNIVERSE, POWER SPECTRUM

The operator for fluctuations is next continued inside region I to find its effect in the open universe. For $u > u_R$, approaching the light cone,

$$\begin{aligned} \phi^+_p(u,\tau)|_{u>u_R} &= (b_{+,p}(u_R)\alpha_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\beta_{-p})B_p(u,\tau) \\ &+ (b_{+,p}(u_R)\beta_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\alpha_{-p})B_{-p}(u,\tau) \end{aligned} \tag{5.1}$$

At the light cone, $\sigma \to 0$ ($u \to \infty$). Inside region I, $\sigma = -iT, \tau = \chi - i\pi/2$, and time and space get interchanged. The analogue of u in region II is the conformal coordinate η, obeying $e^\eta = \tanh(T/2)$.

The σ, τ coordinate system becomes singular at the light cone. The correspondence of wavefunctions across the light cone was found in [2], and can be taken over directly, as they emphasize, since only the short distance behavior of the fields is used:

$$\begin{aligned} & e^{-ipu}\cosh u \tfrac{e^{ip\tau}}{\cosh\tau} \to (+i)\tfrac{\sin p\chi}{\sinh\chi} e^{(ip-1)\eta} \\ & e^{-ipu}\cosh u \tfrac{e^{-ip\tau}}{\cosh\tau} \to 0 \\ & e^{ipu}\cosh u \tfrac{e^{ip\tau}}{\cosh\tau} \to 0 \\ & e^{ipu}\cosh u \tfrac{e^{-ip\tau}}{\cosh\tau} \to (-i)\tfrac{\sin p\chi}{\sinh\chi} e^{-(ip+1)\eta} \end{aligned} \tag{5.2}$$

Only half of the modes propagate inside the light cone for massless fields, the other propagate along the light cone instead [2]. Since the matching depends only on the coefficients of $i\tau$ in the exponential, these rules can be used directly to match the modes for general ℓ if desired.

The basis functions then become: (one can think of u here as $u + \delta$ in the earlier basis functions)

$$\begin{aligned} B_p(u,\tau) \quad &\to_{u\to\infty} -iH_T \tfrac{1}{4\pi\sqrt{p}} \tfrac{e^{\frac{\pi p}{2}}}{\sqrt{e^{\pi p} - e^{-\pi p}}} e^{(-ip-1)\eta} \tfrac{\sin p\chi}{\sinh\chi} \equiv \tilde{B}_p(\eta,\chi) \\ B_{-p}(u,\tau) \ &\to_{u\to\infty} -iH_T \tfrac{1}{4\pi\sqrt{p}} \tfrac{e^{\frac{-\pi p}{2}}}{\sqrt{e^{\pi p} - e^{-\pi p}}} e^{(ip-1)\eta} \tfrac{\sin p\chi}{\sinh\chi} \equiv \tilde{B}_{-p}(\eta,\chi) \ . \end{aligned} \tag{5.3}$$

With this continuation across the future light cone of the bubble center, eq. (3.22) becomes

$$\begin{aligned} \delta\hat{\phi}(\eta,\chi) = \ & \textstyle\int_0^\infty dp \ \Big[(b_{+,p}(u_R)\alpha_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\beta_{-p})\tilde{B}_p(\eta,\chi) \\ & + (b_{+,p}(u_R)\beta_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\alpha_{-p})\tilde{B}_{-p}(\eta,\chi)\Big] \, \hat{a}^+(+p) \\ & + \Big[(b_{+,p}(u_R)\alpha_{-p} - \tfrac{D_2(-p)}{|D_2|} b_{-,p}(u_R)\beta_p)\tilde{B}_{-p}(\eta,\chi) \\ & + (b_{+,p}(u_R)\beta_{-p} - \tfrac{D_2(-p)}{|D_2|} b_{-,p}(u_R)\alpha_p)\tilde{B}_p(\eta,\chi)\Big] \, \hat{a}^+(-p) \\ & + h.c. \end{aligned} \tag{5.4}$$

(Recall there are no discrete states for the values of M^2 being considered.)

This analytically continued wavefunction can be verified to have proper normalization, providing a check on the calculation of normalization given earlier. The future in de Sitter space has two light cones which must be included, region I, where the open universe interior to the bubble is found, and that of the antipode of the center of the bubble

(for descriptions see, *e.g.* [2,31,34]), region III. (These are regions L, R in the notation of [31].) For our purposes, continuing into region III can be done by using symmetry and then verifying that a normalized wavefunction in region II remains normalized when continued into region I, III using these rules. The modes continued into region III are exterior to the bubble wall, and consequently have mass corresponding to the false vacuum mass. As time and space are interchanged in these forward light cones, the Klein Gordon inner product now involves Wronskians of the analytically continued u dependent part of the wavefunctions. The inner product is calculated on a fixed time surface in both forward light cones and then added. Although, as stated before, the fixed time surfaces are not Cauchy surfaces, the wavefunctions under consideration are the same ones (restricted to each light cone) discussed in [31]. They do not correspond to supercurvature modes and consequently have sufficiently fast falloff at infinity that their inner products on the fixed time surface inside the light cones correspond to their inner products on a Cauchy surface. It is also useful to know, in order to show this, that $|\alpha_p|^2 - |\beta_p|^2 = a^2_{\nu'+} - a^2_{\nu'-}$ which can be shown with the definitions in eq. (4.4) and properties of Wronskians.

The fluctuation $\delta\hat{\phi}$ inside the light cone sources gravity, and can be matched directly onto fluctuations of the gravitational potential Φ using results of [2], as follows. Since the fluctuations are small, they can be described by expanding the metric around de Sitter space and the field ϕ around its classical value, as was done in [2], also see [22] for another discussion. The result is that a source $\phi \approx e^{(\pm ip-1)\eta}$ corresponds to a "gauge invariant gravitational potential"

$$\Phi \approx \frac{4\pi G V_{,\phi}}{\pm ip + 2} e^{(\pm ip+1)\eta} \; . \tag{5.5}$$

Then, as [2] show, when H remains constant and the potential is linear, this matches onto the exact solution:

$$\Phi = \frac{4\pi G V_{,\phi}}{\pm ip + 2} e^{(\pm ip+1)\eta} [1 - \frac{p \pm i}{3(p \mp i)} e^{2\eta}] \; . \tag{5.6}$$

For the $\delta\hat{\phi}$ found here, the corresponding positive frequency $\hat{\Phi}$ is:

$$\begin{aligned} \hat{\Phi}(\eta,\chi) \; &= -i\tfrac{4\pi G V_{,\phi}}{4\pi H_T} e^{\eta} [\textstyle\int_0^\infty \frac{dp}{\sqrt{p}} \frac{\sin p\chi}{\sinh\chi} \frac{1}{\sqrt{2\sinh p\pi}} \\ &\{(e^{\frac{\pi p}{2}}(b_{+,p}(u_R)\alpha_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\beta_{-p}) \tfrac{1}{2-ip} e^{-ip\eta}[1 - \tfrac{p-i}{3(p+i)} e^{2\eta}] \\ &+ e^{\frac{-\pi p}{2}}(b_{+,p}(u_R)\beta_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\alpha_{-p}) \tfrac{1}{2+ip} e^{ip\eta}[1 - \tfrac{p+i}{3(p-i)} e^{-\eta}])\hat{a}(p) \\ &+ (e^{\frac{-\pi p}{2}}(b_{+,p}(u_R)\alpha_{-p} - \tfrac{D_2(-p)}{|D_2|} b_{-,p}(u_R)\beta_p) \tfrac{1}{2-ip} e^{ip\eta}[1 - \tfrac{p+i}{3(p-i)} e^{2\eta}] \\ &+ e^{\frac{\pi p}{2}}(b_{+,p}(u_R)\beta_{-p} - \tfrac{D_2(-p)}{|D_2|} b_{-,p}(u_R)\alpha_p) \tfrac{1}{2-ip} e^{-ip\eta}[1 - \tfrac{p-i}{3(p+i)} e^{2\eta}])\hat{a}(-p)\} \end{aligned} \tag{5.7}$$

The power is defined by taking

$$\langle \Phi(\chi = 0, \eta)\Phi(\chi, \eta)\rangle = \int_0^\infty dp\, p \frac{\sin p\chi}{\sinh\chi} P_\Phi(p, \eta) \; + \textit{discrete states} \tag{5.8}$$

and isolating the growing mode by taking $\eta \to 0^-(T \to \infty)$. This gives

$$\begin{aligned} P_\Phi(p) = \; &(G V_{,\phi})^2 \tfrac{1}{p(1+p^2) 2\sinh\pi p} \tfrac{4}{9H_T^2} \\ &\{|e^{\frac{\pi p}{2}}(b_{+,p}(u_R)\alpha_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\beta_{-p}) \\ &- \tfrac{p+i}{p-i} e^{\frac{-\pi p}{2}}(b_{+,p}(u_R)\beta_p - \tfrac{D_2(p)}{|D_2|} b_{-,p}(u_R)\alpha_{-p})|^2 \\ &+ |(e^{\frac{-\pi p}{2}}(b_{+,p}(u_R)\alpha_{-p} - \tfrac{D_2(-p)}{|D_2|} b_{-,p}(u_R)\beta_p) \\ &- \tfrac{p-i}{p+i} e^{\frac{\pi p}{2}}(b_{+,p}(u_R)\beta_{-p} - \tfrac{D_2(-p)}{|D_2|} b_{-,p}(u_R)\alpha_p)|^2 \\ &= (G V_{,\phi})^2 \tfrac{1}{p(1+p^2) 2\sinh\pi p} \tfrac{4}{9H_T^2} [e^{\pi p} + e^{-\pi p} \\ &- \tfrac{1}{D_1^2 - |D_2|^2} \{\tfrac{p+i}{p-i}(\alpha_{-p}\beta_p - \beta_p^2 \tfrac{C_2^*}{2C_1} - \alpha_{-p}^2 \tfrac{C_2}{2C_1}) + \tfrac{p-i}{p+i}(\alpha_p\beta_{-p} - \beta_{-p}^2 \tfrac{C_2}{2C_1} - \alpha_p^2 \tfrac{C_2^*}{2C_1})\}] \end{aligned} \tag{5.9}$$

Here, the identity

$$\begin{aligned} &D_1^2 - |D_2|^2 = D_1 - \tfrac{C_2^* D_2 + C_2 D_2^*}{2C_1} \\ &= \tfrac{1}{2}(1 + |\alpha_p|^2 + |\beta_p|^2 - \tfrac{C_2^*}{C_1}\alpha_p\beta_p - \tfrac{C_2}{C_1}\alpha_{-p}\beta_{-p} - |\tfrac{C_2}{C_1}|^2) \end{aligned} \tag{5.10}$$

was used.

Taking $R \to 0$, *i.e.* $\alpha_p \to a_{\nu'+}$, $\beta_p \to -\frac{C_2}{|C_2|} a_{\nu'-}$, this reduces to the case found in [20]:

$$(GV_\phi)^2 \frac{4}{9H^2} \frac{1}{p(p^2+1) 2\sinh \pi p} \left[e^{\pi p} + e^{-\pi p} + \left(\frac{C_2}{C_1} \frac{p+i}{p-i} + \frac{C_2^*}{C_1} \frac{p-i}{p+i} \right) \right] . \tag{5.11}$$

Generally $|\frac{C_2}{C_1}| \leq 1$, so in the $R = 0$ case, this means that the p dependence is in an envelope between $\coth \frac{\pi p}{2}$ and $\tanh \frac{\pi p}{2}$, going to $(p(p^2+1))^{-1}$, the scale invariant spectrum, approximately when $p \geq 2$.

For $0 \leq p \leq 2$ a wide range of values are possible, depending upon the specific values of $\delta, R, M/H$. For $p \geq 2$, when the radius is nonzero, the last term (the last line) can only be significant if it grows faster than the prefactor $\sinh \pi p^{-1}$ goes to zero, in such a way as to be comparable to the first term $e^{\pi p} > 250$.

The magnitude of this term can be bounded as follows. It can be rewritten as

$$4Re[e^{i\phi} \frac{p+i}{p-i} \frac{yA - \frac{\epsilon_c}{2}[1+y^2A^2]}{(\sqrt{1-\epsilon_c^2}(y^2-1) + y^2 + 1)A - y\epsilon_c(A^2+1)}] \tag{5.12}$$

where

$$\begin{aligned} \epsilon_c &= |\tfrac{C_2}{C_1}| \\ A &= \tfrac{C_2}{|C_2|} \tfrac{\beta_{-p}}{|\beta_{-p}|} \tfrac{\alpha_{-p}}{|\alpha_{-p}|} \\ y &= \tfrac{|\alpha_p|}{|\beta_p|} \end{aligned} \tag{5.13}$$

and it has been used that $|\alpha_p|^2 - |\beta_p|^2 = \sqrt{1-\epsilon_c^2}$ and that $\frac{C_2}{C_1} = \frac{C_2}{|C_2|}\epsilon_c$. (Also recall that $\alpha_{-p} = \alpha_p^*$, $\beta_{-p} = \beta_p^*$.) The phase ϕ is real, $e^{i\phi} = \frac{\beta_p}{|\beta_p|} \frac{\alpha_{-p}}{|\alpha_{-p}|}$, and will not be of interest here since the magnitude will be shown to be bounded and small. From their definitions above, one has $0 \leq \epsilon \leq 1$, $1 \leq y$ and $|A| = 1$.

It is possible to extremize the above expression in A and y because the extrema of a ratio of two quadratic polynomials is given by a quadratic equation and can be solved analytically. However, this is of limited use. In the space spanned by physical values of ϵ_c, y, A, the expression was not found to ever be larger than two in absolute value (fixing one parameter, plotting for the full range of the other two, and changing the parameter that was originially fixed). In the limiting value of $\epsilon_c = 1$, the magnitude can be seen directly to be two and when $\epsilon_c \to 0$ it is of magnitude less than two. For any mass value, as p is changed, one runs through a set of allowed values of ϵ_c, y, A, but for $p \geq 2$ this term is effectively irrelevant, and the power spectrum becomes scale invariant to a high degree of accuracy. That is, if equation 5.12 is less than or equal to two in magnitude, then the quantity in brackets in equation 5.9 is between $\coth \pi p/2$ and $\tanh \pi p/2$, just as in the zero radius case.

The effects from the vacuum fluctuations propagating through the bubble wall must be combined with other fluctuations caused by the presence of the wall. In quantizing the field theory around the classical bubble solution $\phi_b(\sigma)$ in flat space, there are two special modes arising, which remain even in the thin wall limit [21]. One is a translational zero mode, $\partial_\mu \phi_b(\sigma)$, where $\partial_\mu = \partial_{x_i}$, and and one is a mode coming from shifting the radius, $\partial_\sigma \phi_b(\sigma)$. The latter state was identified by [11] as having $p^2 = -4$ in the basis functions of eq. (3.6), and appears also in the de Sitter case if one assumes that H changes negligibly during tunneling [12]. The former corresponds [11] to (nonorthogonalized) $\ell = 0, 1$ modes for this value of p. As the explicit form of both is related to derivatives of the classical background, which only varies significantly on or near the wall, they tend to be localized on the wall, and their degrees of freedom can be related to fluctuations of the wall. Although localized, fluctuations in the wall can, at later times, change the definition of where the 'time' is zero. So these fluctuations can produce an asymmetry in space now, since different regions would be at slightly different times in their evolutions. In [7,4,10,12] the bubble wall fluctuations are discussed and contributions to asymmetries and to temperature anisotropies in the CMB calculated.

VI. CONCLUSIONS

In this paper the results of [2,20] were extended to include a finite wall radius and small vacuum energy difference before and after tunneling. The power spectrum appears to generally go over at $p \sim 2$ to an approximately scale invariant spectrum just as in the zero radius case. As most of the discriminating effects found here are at very large scales, cosmic variance may limit their observability. The scale invariant spectrum that seems to arise at smaller

scales (no exceptions were found and perhaps it can be shown rigorously[**]) means that varying the parameters R, δ and false vacuum mass M over a large range for thin wall bubble models gives a generic prediction.

For a given model, these effects can be translated into effects in the cosmic microwave background by combining them with the fluctuations from the wall modes and a model for matter such as cold dark matter. Modifications of this simplified model (wall thickness, tilt in the final spectrum) should be considered, again to quantify the sensitivity of the models to variations in parameters. Also, so far, the gravity wave perturbations have only been calculated for the wall fluctuations themselves [12].

Along other lines, it would be interesting to see what sorts of potentials naturally arise in particle theory models.

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[1] A. Guth, Phys. Rev. D 23(1981)347; A. Guth and E. Weinberg, Nucl. Phys. B212(1983) 321
[2] M. Bucher, A. Goldhaber, N.Turok, Phys. Rev. D52(1995), 3314
[3] K. Yamamoto, M. Sasaki, T. Tanaka, ApJ. 455 (1995) 412
[4] A. Linde, Phys. Lett. B351(1995) 99
[5] L. Amendola, C. Baccigalupi, F. Occhionero, Reconciling Inflation with Openness, astro-ph/9504097(1995)
[6] N. Cornish, D. Spergel, G. Starkman, "Does Chaotic Mixing Facilitate $\Omega < 1$ Inflation?", CRWU-P16-95, astro-ph/9601034
[7] A. Linde, A. Mezhlumian, Phys. Rev. D52 (1995) 6789
[8] S. Coleman, Phys. Rev. D15(1977)2929; 16(1977)1248(E); C.G.Callan and S. Coleman, Phys. Rev. D 16 (1977) 1762; S. Coleman and F. DeLuccia, Phys. Rev. D21(1980) 3305
[9] J. R. Gott III, Nature 295 (1982) 304; J.R.Gott and T. Statler, Phys. Lett. 136B(1984)157
[10] J. Garcia-Bellido, 'Density Perturbations from Quantum Tunneling in Open Inflation', SUSSEX-AST 95/10-1, astro-ph/9510029
[11] T. Hamazaki, M. Sasaki, T. Tanaka, and K. Yamamoto, Phys. Rev. D53(1996) 2045
[12] J. Garriga, "Bubble fluctuations in $\Omega < 1$ inflation" gr-qc/9602025
[13] T. Banks, C.M. Bender, T.T.Wu, Phys. Rev. D8(1973)3346; J. L. Gervais, B. Sakita, Phys. Rev. D16(1977) 3507
[14] V. Rubakov, Nucl. Phys. B245(1984) 481
[15] H. Kandrup, Phys. Lett. B225(1989)222; Phys. Rev. D 42(1990) 3401
[16] T. Vachaspati, A. Vilenkin, Phys. Rev. D43 (1991) 3846
[17] T. Tanaka, M.Sasaki, K. Yamamoto, Phys. Rev.D49(1994) 1039; M. Sasaki, T. Tanaka, K. Yamamoto, J. Yokoyama, Prog. Theor. Phys. 90(1993) 1019; Phys. Lett. B317(1993) 510
[18] T. Tanaka and M. Sasaki, Phys. Rev. D 50 (1994), 6444, Prog. Theor. Phys. 88(1992), 503
[19] K. Yamamoto, T. Tanaka, M. Sasaki, Phys. Rev. D 51(1995) 2968
[20] M. Bucher, N. Turok, Phys. Rev. D52(1995),5538
[21] J. Garriga and A. Vilenkin, Phys. Rev. D44 (1991), 1007; J. Garriga and A. Vilenkin, Phys. Rev. D45(1992), 3469
[22] E.F. Bunn, K. Yamamoto, "Observational Tests of One-Bubble Open Inflationary Cosmological Models", KUNS-1357, CfPA-95-TH-16, ApJ 464 in press
[23] D. Lyth, E. Stewart, Phys. Lett. B252 (1990) 336; B. Ratra, P.J.E. Peebles, Phys. Rev. D52 (1995) 1837; Astrophys. J.Lett. 423 (1994) L5 (1995) astro-ph/9508090
[24] K. Yamamoto, M. Sasaki, T. Tanaka, "Quantum fluctuations and CMB anisotropies in one-bubble open inflation models", astro-ph/9605104

[**]The corresponding expression in [24] seems to be within the bounds conjectured here.

[25] More recent treatments can be found in textbooks also, for example, see R. J. Rivers, Path integral methods in quantum field theory, Cambridge Univ. Press, 1987, Cambridge
[26] L.G.Jensen, P.J. Steinhardt, Nucl. Phys. B237(1984) 176
[27] A.S. Goncharov and A.D. Linde, Sov. J. Part. Nucl. 17(1986) 369; A. Linde, Nucl. Phys. B372(1992) 421
[28] S. Parke, Phys. Lett. 121B (1983) 313
[29] S.W.Hawking and I.G.Moss, Phys. Lett. 110B(1982)35
[30] A. Linde, Particle Physics and Inflationary Cosmology, Harwood Academic Publishers (1993:USA)
[31] M. Sasaki, T. Tanaka, K. Yamamoto, Phys. Rev. D51(1995)2979
[32] S. Hawking, G.F.R. Ellis, The Large Scale Structure of Spacetime (Cambridge:Cambridge University Press,1973)
[33] D. Lyth, A. Woszczyna, Phys. Rev. D52 (1995)3338; David H. Lyth, 'The Grishchuck-Zeldovich Effect in the open universe', LANC-TH9512; astro-ph/9501113; J Garcia-Bellido, A. Liddle, D. Lyth and D. Wands, Phys. Rev. D52(1995) 6750
[34] B. Allen, Phys. Rev. D51 (1985) 3136